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Modeling experiments using quantum and Kolmogorov probability

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Abstract

Criteria are presented that permit a straightforward partition of experiments into sets that can be modeled using both quantum probability and the classical probability framework of Kolmogorov. These new criteria concentrate on the operational aspects of the experiments and lead beyond the commonly appreciated partition by relating experiments to commuting and non-commuting quantum operators as well as non-entangled and entangled wavefunctions. In other words the space of experiments that can be understood using classical probability is larger than usually assumed. This knowledge provides advantages for areas such as nanoscience and engineering or quantum computation.

1. Introduction

Quantum mechanics, taken as a probability theory, presents us with a number of results that appear to be mysterious when discussed in terms of a classical probability theory such as the framework of Kolmogorov. Nevertheless, as quantum concepts penetrate engineering and technology, it certainly is interesting to investigate the applicability of classical probability. Modern nanostructure research does present us with the desire to explain to students the possible limitations to apply the probability theory that they have learned (e.g. from the work of Shannon in courses of information theory) and what type of experiments can and have to be understood exclusively by quantum probability. Considering the appearance of 'negative classical probabilities' for well known quantum experiments [1], this seems to be a tall order. Detailed reflection reveals, however, that the difficulties arise mainly from interpretational issues and not from the mathematical theories themselves. Both quantum probability as formulated by von Neumann and Kolmogorov's probability are well proven frameworks with well defined relations to actual experiments.

Quantum mechanics relates to the experiments by both preparation of a quantum entity and by the actual measurement of that entity with some equipment resulting in an indicator reading. Kolmogorov's probability theory relates to the experiments by the definition of a probability space. The author found that most of the difficulties in the use of both frameworks disappear with the introduction of different probability spaces as follows. We certainly need to admit different probability spaces in the Kolmogorov framework when we describe different preparations of a quantum entity. However, we also need to admit different probability spaces for different macroscopic arrangements of the measurement equipment. While this latter requirement is natural from the modern point of view that the results of the measurements are also determined by quantum processes in the measurement equipment, it does not agree with the usual convention of describing the equipment by some idealization of perceptions of the observer i.e. by idealized sense impressions. For example, a polarizer is just described by some unit vector and rotation of the polarizer just rotates the vector, nothing else. Yet it is clear that the polarizer has to be seen in a consistent theory as a many body system that interacts with the measured quantum entity and may by itself require probability concepts to describe this interaction effectively. Consider the scattering of electrons by two neighboring nanotubes (the 'positive' of the two slit experiment). Even if we think of the tubes only as coupled classical antennas, the scattering must be described by a Kolmogorov space very different from that obtained by adding the effects of two single nanotubes. A great discussion of probabilities for the two slit experiment from a purely mathematical view has been given by Khrennikov [2]. It is the thesis of this paper that the postulate of different Kolmogorov probability spaces for both different preparations and different macroscopic measurement configurations provides a means of using both quantum and Kolmogorov probability for a given science problem without contradictions. This fact, although it may at first glance raise some doubts, becomes trivial as soon as one realizes that naturally we can explain any measurement of a pointer reading of any given experiment by use of a specially constructed probability space. However, the advantage of Kolmogorov's framework (such as the use of all its theorems) can be enjoyed only if we can describe large sets of experiments and measurements on one common (though abstract) probability space. Therefore, in order to use the advantages of Kolmogorov probability, we need to know when we can concatenate any sequence of quantum experiments on one common Kolmogorov probability space. The answer to this question is, unfortunately, mathematically so complex that it goes beyond the partition into experiments corresponding to commuting and non-commuting operators and involves combinatorial-topological concepts. Because the author wishes to present this paper in a volume to honor the experimental work of Dr G Bauer, he has attempted a presentation that will appeal to experimentalists with the hope that he has not compromised with respect to mathematical rigor.

The reader who wishes to penetrate deeper is referred to the general references [1, 3] and is also assumed to have some familiarity with the work of Bell [4] and experiments of quantum optics and Einstein–Podolsky–Rosen (EPR) [5] types of experiments such as that of Aspect and others [6]. Basic to the development of the mathematical point of view presented here is the work of Fine [7], Pitovsky [8], Khrennikov [9] and Hess and Philipp [10] and especially the work of Vorob'ev [13] that actually precedes that of Bell [11] and contains the very complicated combinatorial-topological basis for our main results.

2. Combining quantum and Kolmogorov probability theory

We define in this section, by use of operational concepts (see e.g. [14]), sets of experiments that can be modeled by both quantum and Kolmogorov probability (QK sets). Our emphasis is on the experiments themselves and we avoid to use notions such as commutation or non-commutation in the definition of QK sets because such notions would give 'preference' to the axioms of the quantum probability framework. Only operational concepts are used. This means we attempt to use only the relationship to the actual experiments that both probability frameworks demand. The possibility of describing certain experiments by both probability frameworks can be found in textbooks (see e.g. [1, 15]). However, quantum entanglement and non-commutations are usually presented as show-stoppers for the use of the combined framework. We attempt here to present clear borderlines and we show that certain sets of experiment can still be modeled *a la* Kolmogorov, in spite of the fact that their quantum model involves entanglement and non-commuting quantum operators.

The use of the word probability amounts to an admission that there are some phenomena involved that we can not easily control and influence or even understand in principle. Nevertheless we need to link these phenomena to actual experiments to logically deal with them. We therefore need to introduce an element into the respective theory that is linked to the elements of the 'real world' (if this expression is permitted) but has no completely defined meaning in that real world. The quantum wavefunction has no direct meaning in terms of sense impressions (see e.g. [1, 15]) and the same is true for the elements of the abstract Kolmogorov sample space of which one must be chosen by Tyche (the goddess of fortune) to 'crystallize' something 'into existence' [3]. Our main point is that we endow the probability space with a general dependence on both preparation of quantum particles and macroscopic equipment arrangements.

2.1. Consistent random variables and probability spaces

The sample space Ω [3] relates Kolmogorov's framework to the actual experiments and represents the set of all possible outcomes of the experiments in a mathematical way. It is abstract and general but must correspond to all the Machian sense impressions related to the experiments in a logical way and without contradictions in order to make probability theory a scientific tool, a 'pre-statistics'.

Consider, for example, photons propagating toward a polarizer. If the photon traverses the polarizer we say that we have measured a '1' if we do not measure a thing we say we have a '0'. Thus the possible outcomes are $0, 1$. However, each of the experiments involves many factors. A physical description of a given experiment includes the actual propagation of a photon, the geometrical arrangement of the polarizer(s), the temperature of the equipment and other factors that may or may not be known. To consider all of these factors in a logical fashion we need to relate experiments to abstract 'indecomposable' elements ω [16] of the sample space Ω (also referred to as 'elementary events' [1]). An actual outcome is then 'crystallized into existence' by Tyche's choice of a particular element of the set Ω that is often denoted by ω^{act} [3]. We also wish to include into our considerations composite experiments i.e. experiments performed in many stages e.g. sequences of two coin tosses with the possible outcomes HH, HT, TH, TT where H stands for head and T for tail. Things become demanding if we deal with possible 'hidden' effects such as magnetic substances in the coins and hidden magnets that influence them. Then a very careful treatment of the sample space is necessary to avoid contradictions.

We consider here, for reasons of physical clarity, only countable sets Ω. To complete a Kolmogorov model, further steps are needed. One introduces a probability measure $P(\Omega) = 1$ which assigns to each event (subset of Ω) a real number of the interval $[0, 1]$. We then have formed a probability space (Ω, P). Random variables are real valued functions on that probability space. We may link the random variables, and do so below, to the possible real valued experimental outcomes such as the discrete (because of assumed countability) eigenvalue spectrum of certain quantum operators and corresponding experiments. For example, we may label the random variables by some physical property of a polarizer or Stern–Gerlach magnet (and corresponding quantum operator) that is used in a spin related experiment and select the range of the random variable (function) corresponding to the two possible spin eigenvalues. This is what is usually done. We add here the possibility to change the probability space on which the random variable is defined if we change macroscopic settings e.g. the direction of the magnets.

It is important to note that in the Kolmogorov framework Tyche must choose from one given probability space (Ω, P) and that her chosen ω^{act} must correspond to the known facts of a given experiment such as the propagation of a photon toward a polarizer with given setting. If we consider experimental sets with very different experimental arrangements and in addition construct certain given random variables associated with them, then Tyche may not be able to choose consistently from one probability space as will be shown below (see also [17]). Consistency of joint probabilities is, in fact, *a premise* in Kolmogorov's theorem that assures the existence of sets of random variables on one probability space [1]. Therefore one can not postulate that any set of random variables on a probability space exists and describes all experiments we wish to describe. In order to describe different (incompatible) experiments consistently we may have to model them by different sets of random variables and/or probability spaces with different indecomposable elements ω, ω' or ω'', ... and corresponding probability measures $P, P', P'', \ldots$ of the probability spaces (Ω, P), (Ω', P') or (Ω'', P''), ... respectively. This fact is given careful attention in the following.

2.2. Wavefunctions and operators

The operators of quantum mechanics are also related to actual measurements and to the classical physics of the equipment that is used. The results of these measurements are given by indicator readings and correspond to real numbers [14]. The operators act on quantum states, represented by wavevectors $|\psi\rangle$ or wavefunctions. The quantum states themselves can not be defined by use of classical physics and have no direct physical meaning. Only their 'preparation' is described in classical physics terms i.e. operationally by the use of macroscopic instruments. Thus, the quantum framework distinguishes in its definitions and subsequent mathematical use between experiments that are related to a preparation of a quantum 'entity' and the measurements that assign an indicator reading after measuring that entity [14]. This distinction suggests in the language of section 2.1 the involvement of different sample spaces that somehow need to be united if we wish to describe both preparation and measurement. We will see below that, under certain conditions, we can attribute a 'collection' of probability spaces to a prepared quantum state. From this collection, a particular probability space is relevant for the measurements that correspond to a given quantum operator (that in turn corresponds to certain equipment settings). The choice of this probability space is, of course, not unique because quantum mechanics does not specify the sample space. The single indicator reading is also not the subject of quantum theory which features only the expectation values of the indicator readings as its main result. In contrast, the Kolmogorov framework introduces random variables that are functions of sample space elements and assume a definite single value for any ω^{act} which, however, in order to introduce probability is chosen by Tyche. The determination of the circumstances that permit consistent definition of random variables corresponding to any number of quantum operators and/or wave functions on one common Kolmogorov space is not straightforward and is addressed in detail in the sections below.

2.3. Quantum Kolmogorov sets of experiments

Here we describe the combination of quantum and Kolmogorov probability for very restricted but nontrivial sets of experiments that we call quantum Kolmogorov (QK) sets. We emphasize what is important for the combined use of quantum and Kolmogorov probability, particularly the proper connection of the probability space and random variables to the actual experiments as well as the possibility of using more than one probability space for sets of distinctly different experiments. What we use from the framework of quantum theory are the concepts of quantum states $|\psi\rangle$ and operators $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}, \ldots$ that act on these states and have real eigenvalues $E_N^{\hat{a}}, E_L^{\hat{b}}, E_S^{\hat{c}}, \ldots$. As mentioned, to avoid mathematical complexities, we assume a countable number of eigenvalues. Then we have $N, L, S = 1, 2, 3, \ldots$. We do include the use of tensor product states and operators as defined in all texts and point the reader to [1] and [15] for details. Before giving further discussions we start with an important operational definition.

Definition of a QK set: Any set of experiments is defined as a quantum Kolmogorov (QK) set if and only if (i) actual measurements correspond to at most two indicator readings of a given measurement configuration that is assembled at the start of all measurements of the set in 'identical' fashion. 'Identical' is defined by the classical physics that affects the instrument indicators, (ii) the entities that are measured are identically prepared in textbook fashion [14] *and correspond to at most two (quantum) particles and (iii) each indicator reading is taken for precisely one prepared (quantum) entity.*

This definition of experiments is typical for definitions in quantum texts and clearly covers the relationship of quantum probability to experiments. The fact that it also covers Kolmogorov probability is contained in the limitation of the numbers of both measurement and preparation to two with fixed measurement settings.

We will show that a QK set of experiments as just defined can always be modeled by the use of both the quantum and Kolmogorov (combined) probability framework. We will also show that the combined framework can be used for certain unions of QK sets that relate to large numbers of random variables as well as quantum operators, even non-commuting operators and entangled wavefunctions, as long as certain experimental and mathematical conditions are fulfilled.

Note that the above definition of a QK set encompasses nontrivial quantum experiments such as the Aspect experiment for one pair of equipment settings. In this experiment, entangled pairs are propagating to the polarizers and their transmission or non-transmission is measured on each side. A set of such experiments is a quantum Kolmogorov set if and only if the polarizers on each side are arranged during the course of each measurement of the entire set in a given way. Between the measurements of the given set other polarizer settings may be chosen. For the Aspect experiment, the quantum operator corresponding to such a set is a tensor product operator of two (different) spin matrices. The quantum entities (entangled pair) are prepared in a Bell state. Of course, any other state(s) corresponding to a well defined preparation of at most two quantum particles are permitted by the above definition. Clearly such experiments can be modeled successfully by quantum mechanics. As we will see, the outcomes of such experiments can also be modeled *a la* Kolmogorov by use of two random variables on one probability space (proven in the lemma below). It was pointed out to the author by Khrennikov that also the well known '*firefly in the box*' (thought) experiments [18] follow the above definition and can be collected into QK sets. These thought experiments have assumed a special importance in the area of quantum logic. Experimental results of interferometers such as that of Michelson and of Mach–Zender can also be collected in QK sets. The proof for this is already contained in texts such as [19] if we only include the postulate that the probability space describing a given experiment changes due to any macroscopic rearrangement of the measurement equipment (e.g. the introduction of an obstacle into the path of an interferometer), and that furthermore a common abstract probability space may not exist for all the possible variations of equipment change.

The union of arbitrary quantum Kolmogorov sets may be but does not have to be a quantum Kolmogorov set. Therefore, we must use, at least in principle, a different sample space and corresponding probability space for the modeling of each different QK set. For example for a set of measurements corresponding to some equipment settings described by the operator $\hat{a}$ operating on any given state $|\psi\rangle$ we define sample space elements ω of the sample space Ω and we define a probability space (Ω, P) with $P(\Omega) = 1$. We also define random variables to describe the possible experimental outcomes for the given equipment settings. The values that these random variables can assume are the eigenvalues of the quantum operator related to the equipment settings. Any possible measurement outcomes of a QK set are then described by a random variable of the form:

$$E_N^{\hat{a}}(\omega) \tag{1}$$

which is a function on the probability space (Ω, P). The quantum number $N = 1, 2, 3, \ldots$ can also be seen as a random variable on the same probability space. Tyche's choice of an ω^{act} crystalizes then the measurement result into existence. For example we may have $E_N^{\hat{a}}(\omega^{\mathrm{act}}) = E_1^{\hat{a}} = +1$. If we change to different experimental settings we introduce a different probability space Ω', P' that now corresponds to a different operator, say $\hat{b}$. We describe the possible experimental outcomes of this set by:

$$E_L^{\hat{b}}(\omega'). \tag{2}$$

The probability space is chosen in such a way as to fulfill the following equation for the quantum expectation value of $|\psi\rangle$ with respect to the measurement set corresponding to the operator $\hat{a}$ [20, 1]:

$$\langle\psi|\hat{a}|\psi\rangle = \int_{\Omega} E_N^{\hat{a}}(\omega) P(\mathrm{d}\omega) \tag{3}$$

and similarly for $\hat{b}$

$$\langle\psi|\hat{b}|\psi\rangle = \int_{\Omega'} E_L^{\hat{b}}(\omega') P'(\mathrm{d}\omega') \tag{4}$$

etc. Here we have used the standard notation of probability theory for Lebesque integrals.

In this way we can subdivide in essence all the experiments related to quantum (as well as classical) physics into sets that can be modeled a la Kolmogorov, however, each on a different Kolmogorov space. The interesting question is, of course, whether unions of these sets can also be modeled on another single Kolmogorov space e.g. a product space. Furthermore there are questions whether we can deal with quantum entanglement that way.

3. Entanglement, Bell and Vorob'ev

The objection can be made that there are problems when attempting to model or even understand quantum mechanics by using random variables on a Kolmogorov space whenever quantum entanglement [15] is involved. Therefore we need to investigate the mathematical basis of the no-go proofs of Bell and others. This will lead us to the definition of closed quantum Kolmogorov (CQK) sets in the next section. Fortunately, much work has been already done to extract this mathematical content and we are basing the following section on the extensive discussion in [10, 20].

3.1. The mathematics related to Bell's no-go proof

Instead of the general $E_N^{\hat{a}}(\omega)$ etc from above we now consider the results of EPR experiments in the notation of Bell [11] i.e. the spin measurement outcomes $A_{\mathbf{a}}((\cdot)), A_{\mathbf{b}}((\cdot)) = \pm 1$ in a measurement station S_1 and $B_{\mathbf{b}}((\cdot)), B_{\mathbf{c}}((\cdot)) = \pm 1$ in a second station S_2 with corresponding tensor products of Pauli spin matrices $\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}, \sigma_{\mathbf{c}}$. Here $(\cdot)$ indicates an element of an appropriate sample space that can be different for different experiments. This represents a slight generalization to Bell's parameter λ representing elements of reality whereas $(\cdot)$ can be chosen by Tyche. One portion of an entangled pair is sent to the respective stations where the results are registered at certain measurement times t_j as measured by a clock in the reference frame of the stations (assumed for our purpose here to be identical) and collected.

We first prove that compatible (i.e. not mutually exclusive) EPR experiments may be concatenated on one probability space and form a QK set. We use, as Bell did, the relation $B_{\mathbf{a}}(\omega) = -A_{\mathbf{a}}(\omega)$ just for the sake of an efficient presentation.

Lemma. *For one given setting in each of the two stations, and thus for a given set of QK experiments performed at measurement times t_j in the given reference frame of the two measurement stations it is possible to find a single abstract* *probability space on which the functions $A, B = \pm 1$ can be defined and to obtain the pair expectation value prescribed by quantum mechanics.*

Proof. The goal is to obtain the quantum mechanical pair expectation $M(A_{\mathbf{a}}(\omega)A_{\mathbf{b}}(\omega))$ [11, 14]:

$$M(A_{\mathbf{a}}(\omega)A_{\mathbf{b}}(\omega)) = -\langle\psi_B|\sigma_{\mathbf{a}} \otimes \sigma_{\mathbf{b}}|\psi_B\rangle = \mathbf{a}\cdot\mathbf{b} \tag{5}$$

where $\otimes$ denotes the tensor product and $|\psi_B\rangle$ is the Bell wavefunction given by:

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right). \tag{6}$$

The following joint probability measure results in $M(A_{\mathbf{a}}(\omega)A_{\mathbf{b}}(\omega))$ of equation (5) as can be found by inspection (see [20]):

$$P(\omega : A_{\mathbf{a}}(\omega) = (-1)^n, A_{\mathbf{b}}(\omega) = (-1)^k) = \tfrac{1}{4}(1 + (-1)^{n+k}\mathbf{a}\cdot\mathbf{b}). \tag{7}$$

Here $k, n = 1, 2$. The probability measure so defined is unique because it is the only measure that also fulfills $M(A_{\mathbf{a}}, \omega) = M(A_{\mathbf{b}}, \omega) = 0$ as required by quantum mechanics [10]. Thus, EPR spin experiments as discussed by Bell and restricted to precisely one setting on each side, i.e. to compatible experiments, can be described by one abstract probability space with elements $\omega \in \Omega$ that represents both source and equipment variables. We have for all probabilities $0 \leqslant P \leqslant 1$ and no contradiction of the Bell type follows. Therefore spin related EPR experiments with entangled pairs do form QK sets and the question raised about them is answered. □

Unfortunately these QK sets can not be arbitrarily concatenated to larger QK sets as can be seen from the following theorem.

Theorem. *Using all the possible magnet settings (with two or more settings in each station) it is impossible to use without contradictions the algebra of three (or more) random variables defined on a common probability space Ω in cyclical algebraic expressions such as*

$$A_{\mathbf{a}}(\omega)B_{\mathbf{b}}(\omega) + A_{\mathbf{a}}(\omega)B_{\mathbf{c}}(\omega) + A_{\mathbf{d}}(\omega)B_{\mathbf{b}}(\omega) - A_{\mathbf{d}}(\omega)B_{\mathbf{c}}(\omega) \tag{8}$$

and to recover at the same time all possible expectation values such as $M(A_{\mathbf{a}}(\omega)A_{\mathbf{b}}(\omega)) = \mathbf{a}\cdot\mathbf{b}$, $M(A_{\mathbf{a}}(\omega')A_{\mathbf{c}}(\omega')) = \mathbf{a}\cdot\mathbf{c}$ and $M(A_{\mathbf{b}}(\omega'')A_{\mathbf{c}}(\omega'')) = \mathbf{b}\cdot\mathbf{c}$ (notice the different sample space elements for each pair).

The proof below illustrates that it is not the expectation value itself (as given by quantum mechanics) that leads to a contradiction. As we have seen from the lemma we can easily create the result $\mathbf{a}\cdot\mathbf{b}$ etc by a classical simulation. The contradiction arises from the involvement of mutually exclusive experiments and the use of more than two random variables in cyclical arrangement.

Proof. We present here only an outline of the major ideas with references to previous work that contain the precise and complete elements of the proof.

It was shown in [10] that Bell's inequalities represent a special case of the more general mathematical framework of Vorob'ev [13] who showed that *'..it is not always possible to construct a vector random variable with given consistent projections.'* Vorob'ev talks about vector random variables with one dimensional variables as components. Vorob'ev's [13] work gives precise mathematical conditions for the validity of his statements and theorems and the serious reader should at least understand theorem 1 of [10] and the first page of [13]. However, the essence is this:

The pairs of random variables $A_{\mathbf{a}}(\omega)A_{\mathbf{b}}(\omega)$, $A_{\mathbf{a}}(\omega)A_{\mathbf{c}}(\omega)$ and $A_{\mathbf{b}}(\omega)A_{\mathbf{c}}(\omega)$ form a 'closed loop' or display some 'cyclic behavior' [13]. Then, once the pair distributions of $A_{\mathbf{a}}(\omega)A_{\mathbf{b}}(\omega)$, $A_{\mathbf{a}}(\omega)A_{\mathbf{c}}(\omega)$ are given one cannot choose that of $A_{\mathbf{b}}(\omega)A_{\mathbf{c}}(\omega)$ with complete freedom and at the same time require that $A_{\mathbf{a}}(\omega)$, $A_{\mathbf{b}}(\omega)$, $A_{\mathbf{c}}(\omega)$ are all random variables defined on one common probability space.

In terms of the algebra of random variables one finds the well known constraints on the possible outcomes for four setting pairs:

$$\Gamma = A_{\mathbf{a}}(\omega)B_{\mathbf{b}}(\omega) + A_{\mathbf{a}}(\omega)B_{\mathbf{c}}(\omega) + A_{\mathbf{d}}(\omega)B_{\mathbf{b}}(\omega) - A_{\mathbf{d}}(\omega)B_{\mathbf{c}}(\omega) = \pm 2 \tag{9}$$

that leads to the Bell type inequality $\Gamma \leqslant 2$. Thus the range (codomain) of the function Γ, which plays a very significant role in the framework of Bell, is restricted to ± 2 because of the assumption of a single domain for the cyclically arranged functions. Otherwise we would have $\Gamma = 0, \pm 2, \pm 4$. Without the cyclicity no contradiction can occur and Kolmogorov probability can be used as shown in great detail and for general topologies by Vorob'ev [13]. This fact originates from the complex demands for the sample space of composite and incompatible experiments. Tyche is simply not able to find a single ω^{act} that fulfills all demands including the appropriate range of functions. Physical explanations (e.g. by quantum non-locality or by Einstein local means) need not be addressed here. □

Summarizing lemma and theorem we can state that experiments involving quantum entanglement can be understood as quantum Kolmogorov (QK) experiments as long as no cyclical arrangement of the random variables is involved.

3.2. Non-commutation of the quantum operators and Nelson's no-go proof

Nelsons theorem asserts that if non-commuting operators are involved there exists in general no one to one correspondence of quantum observables to random variables of the Kolmogorov framework. We have learned above that, as far as EPR experiments are concerned, we have no problems with non-commuting quantum operators as long as the Bell inequalities or the more general Vorob'ev criteria are fulfilled. A violation of the Bell inequalities can and will occur only if non-commuting operators are involved otherwise no Vorob'ev type problem can exist. This would be enough to prove Nelsons theorem. There are, however, other possibilities of the involvement of non-commutation that are also sufficient to validate Nelson's theorem and these are related to measurements that change the quantum state i.e. act like a different preparation of the quantum particles with further measurements involving the changed state. If such 'crosswise' measurements occur then we do not obey the definitions of a QK set. Then the involvement of two non-commuting operators and two random variables linked to these operators may already lead to a contradiction. This can be seen from the following reasoning.

Consider measurements corresponding to the operator combination (quantum observable) $[\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}]/(2\mathrm{i})$ where $[\cdot]$ denotes the commutator and i is the imaginary unit. The problem is now that there exist $|\psi\rangle$ so that the expectation value of $[\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}]/(2\mathrm{i})$ is equal to a positive number δ:

$$\langle\psi|[\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}]/(2\mathrm{i})|\psi\rangle = \delta. \tag{10}$$

This is 'difficult to square' [3], on one common probability space, with the fact that the product of the random variables $A_{\mathbf{a}}(\omega)$ and $A_{\mathbf{b}}(\omega)$ corresponding to $\sigma_{\mathbf{a}}$ and $\sigma_{\mathbf{b}}$ commutes, i.e. $A_{\mathbf{a}}(\omega) \cdot A_{\mathbf{b}}(\omega) = A_{\mathbf{b}}(\omega) \cdot A_{\mathbf{a}}(\omega)$, while $\sigma_{\mathbf{a}}$ and $\sigma_{\mathbf{b}}$ do not. The use of different probability spaces for different experiments avoids also here contradictions because algebra involving $A_{\mathbf{a}}(\omega')$ and $A_{\mathbf{b}}(\omega'')$ makes no mathematical sense.

If we consider only QK sets and exclude crosswise measurements between different QK sets, then we can not incur any operator combinations and corresponding random variables that carry the problems outlined above. Furthermore, for spin related EPR experiments we always have:

$$\langle\psi_B|\sigma_{\mathbf{a}} \otimes \sigma_{\mathbf{b}}|\psi_B\rangle = \langle\psi_B|\sigma_{\mathbf{b}} \otimes \sigma_{\mathbf{a}}|\psi_B\rangle \tag{11}$$

and therefore can not get in any conflict with the relation $A_{\mathbf{a}}(\omega) \cdot A_{\mathbf{b}}(\omega) = A_{\mathbf{b}}(\omega) \cdot A_{\mathbf{a}}(\omega)$.

These facts bring home one of the purposes of the Einstein–Podolsky–Rosen paper which was to investigate situations where such crosswise measurements are not performed and the Uncertainty Principle and the corresponding 'incompatibility' of experiments is therefore not as directly involved. By excluding crosswise experiments EPR did identify a large number of measurements that correspond to non-commuting operators as well as preparations of (entangled) quantum entities and still can be modeled classically i.e. by the Kolmogorov framework. Of course, as we also know now, this can only be done as long as no topological cyclicity of the system of random variables is involved.

The important corollary of this subsection is that if we exclude 'crosswise QK measurements' we do not need to be concerned about clashes of non-commutation with the Kolmogorov framework as long as no closed loop or cyclic arrangement of random variables is involved.

4. Closed quantum Kolmogorov (CQK) sets

From the above discussions it is clear that QK sets can be modeled by the combined quantum and Kolmogorov approach even if quantum entanglement is involved. QK sets contain at most two measurement outcomes and correspondingly at most two random variables. Because two random variables can not form a closed loop, consistency is guaranteed. No problems with non-commutation can occur for QK sets because they correspond only to a given experimental setting and therefore to one (tensor product) quantum operator. We can therefore use all the tools of the Kolmogorov framework. Results of Kolmogorov probability have in fact been used without much justification [3] to assess the statistical significance of the Aspect experiment and other well known experiments that involve entangled wavefunctions. Below we develop and summarize the mathematical and experimental conditions subject to which also unions of QK sets can be modeled by the combined framework. Such a union of QK sets is referred to as a closed quantum Kolmogorov (CQK) set. Closure denotes here the fact that the union of QK sets can be modeled by the rules of the combined framework as they apply to the given set of experiments.

4.1. Well known CQK sets, QK unions that permit use of the combined framework

It is well known that all experiments and therefore also all QK sets corresponding to *commuting operators* and a given wavevector can be modeled on one common probability space because commuting operators have a common set of eigenvectors. We can even form unions of QK sets for different wavevectors and model them *a la* Kolmogorov without contradiction. The reason for this can be found in the fact that neither an argument involving equation (10) nor an argument involving closed loops can be made as long as all operators commute. Commutation is thus certainly a sufficient condition for the possibility of using Kolmogorov probability and therefore for forming CQK sets by forming the union of QK sets. As we will show, however, it is not a necessary condition for certain restricted sets of experiments. It is well known, for example, that no Bell type contradiction can be obtained for EPR experiments that correspond to commuting operators in one wing only, because experiments corresponding to commuting operators can be seen in essence as experiments with identical equipment settings. This means that no closed loop can be involved in this case.

For experiments corresponding to tensor product wavefunctions and tensor product operators of the quantum framework one can, *by assuming independence*, also construct corresponding product probability spaces for Kolmogorov's framework and thus form CQK sets. Therefore no contradictions can be obtained also in this case even if the experiments correspond to different wavefunctions and non-commuting operators because of the assumed independence that permits the formation of product measures on the Kolmogorov side. Crosswise measurements are, of course, not considered here. The experiments covered in this way do not include all possible QK sets because of the restriction to tensor product wavefunctions. The question arises therefore whether operational and/or mathematical conditions exist that permit the use of the combined framework also for unions of QK sets that correspond to non-commuting operators and entangled wavefunctions. A positive answer to this question is given in the next section.

4.2. Closure of unions of QK sets with entanglement: ECQK sets

What we summarize here is that with the provisions concerning the union of QK sets, as they are given in the next definition, also non-commuting quantum operators and entangled wavefunctions may be involved in the unions of QK sets that still can be modeled by the combined framework and we call them entangled closed quantum Kolmogorov (ECQK) sets. By the definition of QK sets and the rules for forming unions as given in the definition below crosswise experiments are automatically excluded.

We define ECQK sets by:

Definition. *Any union of QK sets corresponding to a given entangled wavefunction (to given entangled wavefunctions) is an ECQK set if and only if (i) any questions regarding measurement sequence are either irrelevant or can be accommodated consistently in both the quantum and Kolmogorov framework and (ii) there arise no contradictions involving closed loops of random variables.*

Note that this definition, in contrast to that of QK sets, is not operational. The reason for this is a natural one: a general closure condition must refer to the mathematical rules of both quantum and Kolmogorov probability. The quantum framework is and always has been used to model unions of QK sets. The conditions for which it is possible to model ECQK sets by the combined framework can therefore be deduced from the conditions under which the set of all random variables of the ECQK set can be consistently defined on one common probability space which they can be if no closed loops in the topological sense of Vorob'ev are involved.

For the special case of spin related EPR experiments for which the random variables assume only values ± 1 the work of Bell (and the earlier work of Bass [12]) provide a very practical way to determine the precise limitations for forming a ECQK set of QK experiments. As shown in [10] and outlined above, a set of spin related EPR experiments with random variables as used by Bell can be modeled on one Kolmogorov probability space if and only if no Bell type inequality is violated. We can therefore concatenate the QK sets of spin related EPR experiments that fulfill all Bell inequalities to form an ECQK set. This also completes our existence proof for ECQK sets.

5. Conclusion

Thus we conclude that classical Kolmogorov probability may be used to model problems related to quantum mechanics as long as certain simple operational and mathematical-topological conditions are fulfilled. This permits the advantageous use of an additional proven and extensive framework of probability theory with a firm set-theoretic basis. Our results clearly show that a general distrust of applying the Kolmogorov framework to quantum experiments is not warranted. The advantages of the combined use certainly justify painstaking investigations of the precise borderline between these two frameworks particularly in the areas of nanoscience and technology and quantum optics which are related to quantum computing and cryptography.

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